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THE NON-STATIONARY CALCULATION OF FLOW PAST A RECTANGULAR WING OF LOW ASPECT RATIO

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THE NON-STATIONARY CALCULATION OF FLOW PAST A RECTANGULAR WING OF LOW ASPECT RATIO

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ABSTRACT. A study of the action of an ideal fluid flow on an infinitely slender wing of rectangular planform performing oscillations in the flow. This study shows that the integral equation (derived by Bisplinghoff et al. on the basis of the Biot-Savart relation) describing the vorticity at the wing and in the wake can be solved by a vortex-lattice technique. The non-stationary forces thus obtained yield satisfactory results for the total lifting force, even for a relatively large lattice step. The effect of the wake appears to be substantial only for distances on the order of the chord and may be disregarded when the wavelength exceeds the chord by a factor of 10 or greater. The lift distribution over an oscillating wing of finite aspect ratio is calculated by reducing the integral equation

$$\hat{v}(x,0,z,t) = -\frac{1}{4\pi}\int_{-L}^{L} d\zeta \int_{0}^{\infty} \frac{\hat{\gamma}_x(\xi,\zeta,t)(z-\zeta) - \hat{\gamma}_z(\xi,\zeta,t)(x-\xi)}{[(x-\xi)^2+(z-\zeta)^2]^{3/2}}\, d\xi$$

to a system of algebraic equations with the distributed vortices replaced by concentrated Π-shaped vortices which consider the influence of the vortex sheet.

1. Fundamental Formulas of Non-stationary Theory /82[1]

We will analyze the effect of the flow of an ideal fluid on an oscillating wing of low aspect ratio. Let an infinitely thin wing of rectangular planform be located in the plane $y = 0$ (Figure 1). The velocity of the oncoming flow is directed along the x axis and is equal to U_0. The velocity of the fluid particles near the wing will be designated as $\vec{V} = \{U_0 + u, v, w\}$. We will assume that the perturbations of the flow are slight, i.e. $u, v, w \ll U_0$, and we will limit ourselves to the linear approximation of the equations of hydrodynamics.

If we know the vertical component of the velocity of the wing $\hat{v}(x,0,z,t)$, then, in accordance with non-stationary theory, the vortical intensity on the wing and in the wake is found from the integral equation:

[1] Numbers in the margin indicate pagination in the foreign text.

$$\hat{v}(x, 0, z, t) = -\frac{1}{4\pi} \int_{-L}^{L} d\zeta \int_{0}^{\infty} \frac{\hat{\gamma}_x(\xi, \zeta, t) \cdot (z-\zeta) - \hat{\gamma}_z(\xi, \zeta, t) \cdot (x-\xi)}{[(x-\xi)^2 + (z-\zeta)^2]^{3/2}} d\xi. \quad (1)$$

This equation may be obtained by using the Biot-Savart law, assuming that the vortex sheet on the wing and in the wake determines the normal velocity component on the wing [1].

The components of linear circulation $\hat{\gamma}_x$ and $\hat{\gamma}_z$ (Figure 1) are interrelated by the equation of continuity:

$$\frac{\partial \hat{\gamma}_x}{\partial x} + \frac{\partial \hat{\gamma}_z}{\partial z} = 0. \quad (2)$$

From the equations of hydrodynamics we may express the pressure difference on the plane y = 0 by the component $\hat{\gamma}_z$:

$$\hat{p}(x, z, t)\Big|_{y=+0} - \hat{p}(x, z, t)\Big|_{y=-0} = \Delta\hat{p} = \rho\left[U_0 \hat{\gamma}_z(x, z, t) + \frac{\partial}{\partial t}\int_0^x \hat{\gamma}_z(\xi, z, t) d\xi\right]. \quad (3)$$

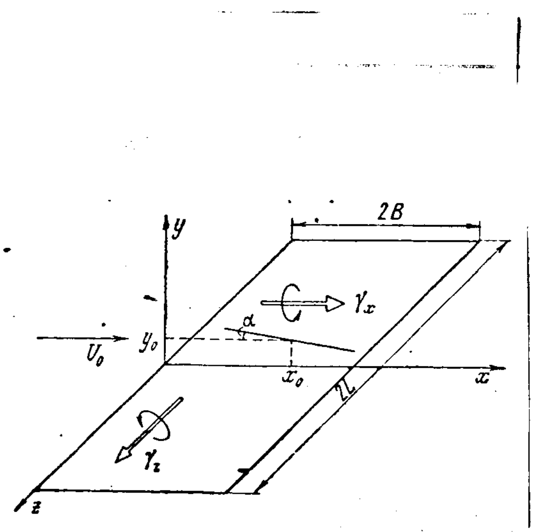

Figure 1

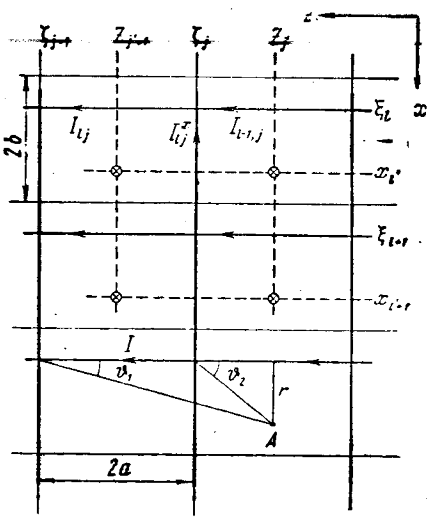

Figure 2

The lifting force acting on some area S of the wing is determined, in turn, /83
by the integral

$$Y = -\iint_S \Delta\hat{p}(x, z, t)\,dxdz. \tag{4}$$

If the motion of the wing is given in the form of the harmonic time function

$$\hat{v}(x, 0, z, t) = v(x, 0, z)e^{i\omega t},$$

equation (3) for amplitudes becomes

$$\Delta p(x, z) = U_0\rho\gamma_z(x, z) + i\omega\rho\int_0^x \gamma_z(\xi, z)\,d\xi. \tag{5}$$

The pressure jumps on the trailing edge of the wing (x = 2B) and in the wake are equal to zero. Therefore, for the range x ⩾ 2B we have

$$\gamma_z(x,z) = -\frac{ik}{B}\Gamma(z)\,e^{-\frac{ik}{B}(x-2B)}, \tag{6}$$

where Γ(z) is the total circulation around the wing:

$$\Gamma(z) = \int_0^{2B}\gamma_z(\xi, z)\,d\xi.$$

Formula (6) takes into consideration the Strouhal number $k = \frac{\omega B}{U_0}$. We can also find the amplitude γ_x in the wake by using the equation of continuity (2). Thus only the function $\gamma_z(\xi, \zeta)$ on the wing is unknown in equation (1). This integral equation can be solved accurately in several particular cases, such as the static problem (k = 0) for wings of infinite and finite span, the non-stationary problem for a wing of infinite span (profile) ($\gamma_z = \gamma_z(x)$, $\gamma_x = 0$).

For the non-stationary problem in the case of a wing of finite span, equation (1) can be solved only approximately.

2. Approximate Method

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Let us now turn our attention from the distribution density of circulation to concentrated vortices. For this purpose we will divide the surface of the wing into nm equal rectangles with the dimensions 2a and 2b (bn = B is the half-chord and am = L is the half-span). In each rectangle, as in [2], we will place a concentrated Π-shaped vortex at a distance b/2 from the leading edge. The ends of the vortex will extend to infinity along the latteral sides of the rectangle (Figure 2).

We will find the intensity of the vortex in rectangle (l,j) $\mathcal{J}_{lj}$ by averaging the density of circulation $\gamma_z(\xi,\zeta)$:

$$\mathcal{J}_{lj} = \frac{1}{2a} \int\limits_{\xi_l - \frac{b}{2}}^{\xi_l + \frac{3b}{2}} d\xi \int\limits_{\zeta_j}^{\zeta_{j+1}} \gamma_z(\xi, \zeta)\, d\zeta \approx 2b\,\bar{\gamma}_{lj}.$$

The coordinates ξ_l and ζ_j are introduced as illustrated in Figure 2; $\bar{\gamma}_{lj}$ is the average density of γ_z in the rectangle. The total circulation around the wing will then be

$$\Gamma_j = \frac{1}{2a} \int\limits_{\zeta_j}^{\zeta_{j+1}} \Gamma(\zeta)\, d\zeta = \sum_{q=1}^{n} \mathcal{J}_{qj}. \tag{7}$$

The analogous averaging of the distribution density of γ_x and the use of the equation of continuity yield, for example, for the total intensity of the vortex $\mathcal{J}'_{lj,1}$ directed along the chord and passing along the right side of rectangle (l,j):

$$\mathcal{J}^{x}_{ij} = \frac{1}{2b} \int\limits_{\xi_l - \frac{b}{2}}^{\xi_l + \frac{3b}{2}} d\xi \int\limits_{\zeta_j}^{\zeta_{j+1}} \gamma_x(\xi, \zeta)\, d\zeta =$$

$$= \frac{1}{2b} \int\limits_{\Delta\xi_l} d\xi \int\limits_{\Delta\zeta_j} d\zeta \int\limits_{0}^{\xi} \frac{\partial \gamma_z(\eta, \zeta)}{\partial \zeta}\, d\eta = 2b \sum_{p=1}^{l} (\bar{\gamma}_{pj} - \bar{\gamma}_{pj-1}).$$

On the other hand, originating from the construction of the Π-shaped vortices (Figure 2), we have for $\mathcal{J}^{x}_{ij}$

$$\mathcal{J}^{x}_{ij} = \sum_{p=1}^{l} (\mathcal{J}_{pj} - \mathcal{J}_{pj-1}).$$

Thus the system described above of concentrated vortices on the wing is equivalent (in total intensity) to the initial distribution surface density.

If we examine the case of the non-stationary flow around a finite wing, we find that vortical intensity is not equal to zero, neither on the surface of the wing nor in the wake. To convert to concentrated vortices in the wake (in contrast to [3]) we will use the very same division into rectangles with the dimensions 2a·2b. The intensity of the Π-shaped vortex in the wake $\mathcal{J}_{\beta j}$ $(\beta = n+1, \ldots, n+N)$ can be expressed through total circulation Γ_j. By using equations (6) and (7) we obtain

$$\mathcal{J}_{\beta j} = \Gamma_j e^{-i\frac{k}{B}\xi_\beta} \left(e^{i\frac{k}{B}\xi_n} - e^{i\frac{k}{B}\xi_{n+1}}\right), \qquad (8)$$ /85

where we make the following definitions (see Figure 2):

$$\xi_n = 2B - \frac{3b}{2}, \qquad \xi_{n+1} = 2B + \frac{b}{2}.$$

The number of rectangles in the wake mN determines the accuracy to which we consider the effect of the wake. It will be shown below that it is sufficient to assume N = 10, since the difference in the values of the theoretical coefficients for N = 5 and N = 10 does not exceed 4%.

It is important to note that the approximate expression (8) is applicable only for k with low values or when the partitioning step is small, i.e. when

$\frac{k}{n} \ll 1$ (furthermore we may still consider the phases of the wave which spreads throughout the wake).

We will now find the expression for local lifting force acting on one rectangle. In accordance with equations (4) and (5) we have

$$Y_{lj} = -\rho U_0 \int\limits_{\Delta\xi_l} d\xi \int\limits_{\Delta\zeta_j} \left[\gamma_z(\xi, \zeta) + \frac{ik}{B} \int\limits_0^{\xi} \gamma_z(\eta, \zeta)\, d\eta \right] d\zeta =$$

$$= -\rho U_0 2a \left[\mathcal{J}_{lj} + i \frac{2k}{n} \left(\sum_{p=1}^{l-1} \mathcal{J}_{pj} + \frac{1}{2} \mathcal{J}_{lj} \right) \right].$$

For simplicity we then introduce the dimensionless circulation

$$I_{lj} = \frac{\mathcal{J}_{lj}}{4\pi U_0 b}. \tag{9}$$

Then the local lifting force will be

$$Y_{lj} = -\frac{\rho U_0^2}{2} \cdot 2a \cdot 2b \cdot 4\pi \left[I_{lj} + i \frac{2k}{n} \left(\sum_{p=1}^{l-1} I_{pj} + \frac{1}{2} I_{lj} \right) \right]. \tag{10}$$

For the concentrated vortices constructed above as in formula (1), we will calculate the vertical component of the perturbed flow in the plane y = 0 by the Biot-Savart law. The velocity field of a section of the linear vortex $\mathcal{J}$ is expressed by the well-known formula:

$$v(A) = \frac{\mathcal{J}}{4\pi r} (\cos\vartheta_1 - \cos\vartheta_2) \tag{11}$$

(see Figure 2 for symbols).

The theoretical values for velocity found from formula (11) will coincide with those found by equation (1) in certain points only. These points, according to the Prandtl hypothesis, will be located in each rectangle at a distance $\frac{3b}{2}$ from the leading edge (denoted by the crosses in Figure 2). The coordinates of these points on the wing are symbolized by $x_{l'}$, $z_{j'}$ (l' = = 1,...,n, j' = 1,...,m).

We will now compute by formula (11) the vertical component of velocity v_{lj} at the point $(x_{l'}, z_{j'})$ produced by one П-shaped vortex I_{lj}:

$$\frac{v_{lj}(x_{l'}, z_{j'})}{U_0} = I_{lj}(A_{lj} - A_{lj+1}),$$

where /86

$$A_{lj} = \frac{-b}{z_{j'} - \zeta_j}\left(\frac{r_{lj}}{x_{l'} - \xi_l} + 1\right), \quad r_{lj} = \sqrt{(x_{l'} - \xi_l)^2 + (z_{j'} - \zeta_j)^2}.$$

By totaling the velocity fields from all vortices distributed on the wing and in the wake we find

$$\frac{v(x_{l'}, z_{j'})}{U_0} = \sum_{l=1}^{n} \sum_{j=1}^{m} I_{lj}\Big[(A_{lj} - A_{lj+1}) + \\ + (e^{i\frac{k}{B}\xi_n} - e^{i\frac{k}{B}\xi_{n+1}}) \sum_{f=n+1}^{n+N} (A_{fj} - A_{fj+1}) e^{-i\frac{k}{B}\xi_f}\Big]. \tag{12}$$

3. Calculation of a Specific Problem

Let us consider one particular problem. Specifically, we will represent the motion **of the wing in** the following form (Figure 1):

$$y(x, t) = y_0 e^{i\omega t + i\varphi} - \alpha(x - x_0) e^{i\omega t},$$

i.e. a rigid wing is displaced vertically as a whole and is simultaneously rotated around the $x = x_0$ axis. Both motions are harmonic with frequency ω and with a phase shift equal to ϕ. Then the amplitude of velocity in each point of the wing will be

$$v(x)|_{y=0} = i\omega[y_0 e^{i\varphi} - \alpha(x - x_0)] - U_0\alpha.$$

By introducing the dimensionless parameters $h = \frac{y_0}{B}$ and $s = \frac{x_0}{B}$ the dimensionless velocity on the straight line $x = x_{l'}$ will be defined by the expression

$$\frac{v(x_{l'})}{U_0} = ik\left[he^{i\varphi} - \alpha\left(\frac{x_{l'}}{B} - s\right)\right] - \alpha. \quad (13)$$

The value for velocity from formula (13) must be substituted into formula (12). The problem of non-stationary flow around a finite wing then reduces to the solution of system nm of linear equations relative to the unknown I_{lj}. This equation system was solved by the "Strela" computer.

A. The static case. We analyzed the static problem, i.e. the case where k = 0, basically for the purpose of checking the effect of the shape and number of rectangles on the calculation results.

We will first of all compare the total lifting force for the various wings. In accordance with (10) the lifting force acting on a wing of area S (S = 2ma•2nb) for k = 0 will be

$$Y_S = -\frac{\rho U_0^2}{2} 2a\,2b\,4\pi \sum_{l=1}^{n} \sum_{j=1}^{m} I_{lj} = -\frac{\rho U_0^2}{2} SC_y;$$

here we have the coefficient of the lifting force

$$C_y = \frac{4\pi}{mn} \sum_{l=1}^{n} \sum_{j=1}^{m} I_{lj}.$$

On the other hand, with a given angle of attack, constant for the entire wing, the coeffieicnt of the lifting force will be proportional to the angle of attack:

$$C_y = \frac{dC_y}{d\alpha} \cdot \alpha.$$ /87

Table 1 lists the experimental results of the coefficient $\frac{dC_y}{d\alpha}$ for various aspect ratios λ = L/B. In the calculation the number of rectangles was varied, both along the span (m) and along the chord (n), and the shape of the rectangles (the ratio $\frac{a}{b}$) was also changed. As seen in the table below, the number and shape of the rectangles had no significant effect on the lifting force (for a given aspect λ). This points out the stability of the algorithm of the problem in relation to the partitioning method. Therefore an increase in the number of rectangles (particularly through the chord) does not improve the accuracy of the calculation for the total lifting force, but does make it possible to obtain a more accurate picture of the distribution of the lifting force along the wing.

TABLE 1

λ	m	n	$\frac{a}{b}$	$\frac{dC_y}{d\alpha}$
0,8	8	5	1/2	1,338
1,6	8	5	1	2,340
2	8	2	1/2	2,707
2	8	3	3/4	2,713
2	8	4	1	2,717
2	6	4	4/3	2,792
2	6	6	2	2,795
2	4	4	2	2,943
3,2	8	5	2	3,535
4,8	8	5	3	4,193
6,4	8	5	4	4,602
8	8	5	5	4,881
∞	(theoretical values)			6,282

The dependence of the magnitude of the coefficient of the lifting force on the aspect is illustrated in Figure 3, where the crosses denote data taken from Table 1; the solid curve represents the results of analogous calculations found in [3] (Figure 12.11, p. 186).

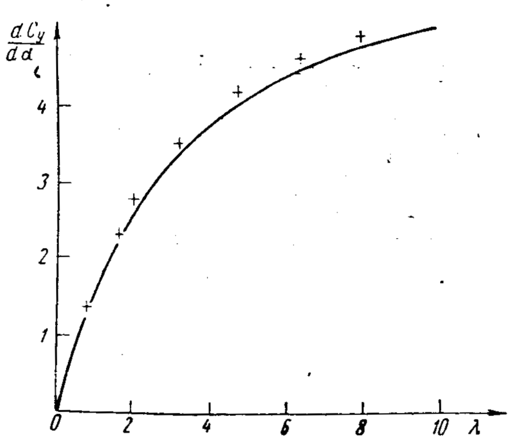

Figure 3

We will now examine the distribution of the lifting force along the span. For this purpose we will compute the total circulation around the wing. By substituting (9) into (7) we obtain for the dimensionless values

$$\frac{\Gamma_j}{U_0 B} = \frac{4\pi}{n} \sum_{q=1}^{n} I_{qj}.$$

This expression may be compared with elliptical distribution of circulation. In accordance with /88
[1], for an elliptical wing of given aspect

$$\frac{\Gamma}{U_0 B} = A\sqrt{1-\left(\frac{z}{L}\right)^2},$$

where

$$A = \frac{\lambda a_0 \alpha}{\lambda + \frac{a_0}{4}} \approx \frac{2\pi\lambda\alpha}{\lambda + \frac{\pi}{2}}.$$

In the latter formula $a_0 \approx 2\pi$.

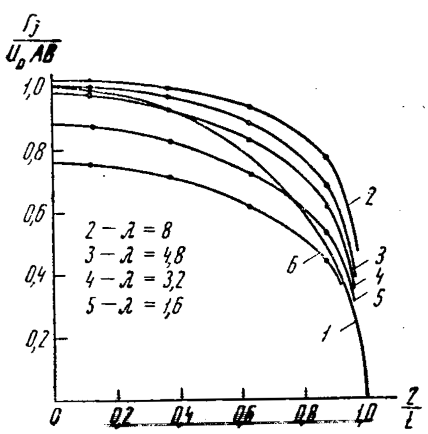

Figure 4

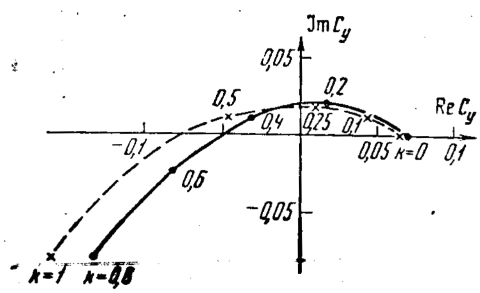

Figure 5

Figure 4 illustrates curve 1, which corresponds to the theoretical elliptical distribution of dimensionless circulation $\frac{\Gamma(z)}{U_0 BA}$, and curves 2-6, which are constructed through points computed for wings of various aspect λ. As might be expected, as λ increases the distribution of the lifting force becomes more uniform. The theoretical results also make it possible to present a clearer picture of the distribution of the lifting force along the surface of the wing.

B. Non-stationary case. In the case where $k \neq 0$ the amplitude of the velocity of the points on the surface of the wing is a complex value. In the concrete case which we are solving (see equation (13)), the real portion of the velocity is the same for the entire wing:

$$\mathrm{Re}\left(\frac{v'}{U_0}\right) = -\alpha - k\sin\varphi,$$

and the imaginary portion is a function of the coordinate $x_{l'}$ along the chord:

$$\operatorname{Im}\left(\frac{v}{U_0}\right) = k\left[h\cos\varphi - \alpha\left(\frac{x_{l'}}{B} - s\right)\right].$$

The problem of calculating the vortices I_{lj} is thus broken down into two problems, i.e. the determination of Re I_{lj} and Im I_{lj}. In this case the terms corresponding to the effect of the wake will play a significant role in equation (12).

Local lifting force, according to (10) will also have a real portion, i.e. in the phase (or opposite phase) with angle of attack α, and an imaginary portion, which is shifted by $\frac{\pi}{2}$ relative to α.

Table 2 lists the values of the coefficient of the lifting force: /89

$$C_y = \frac{Y}{\frac{\rho U_0^2}{2} S} = -\frac{4\pi}{mn}\left[\sum_{l=1}^{n}\sum_{j=1}^{m}\left(I_{lj} + i\frac{2k}{n}\sum_{p=1}^{l-1} I_{pj} + i\frac{k}{n} I_{lj}\right)\right].$$

The calculations were made for the following values of the parameters: α = 0.03; h = 0.1; s = 0.5; aspect ratio of the wing λ = 1.6.

TABLE 2

k	N	Re $C_y \cdot 10^4$	Im $C_y \cdot 10^4$
0,2	0	205	223
0,2	5	205	205
0,2	10	204	204
0,8	0	—1641	—783
0,8	5	—1335	—839
0,8	10	—1317	—805

The table shows two groups of problems for k = 0.2 and k = 0.8. During calculation the number N of rectangles in the wake was varied. We see from the table that when k is small the effect of the wake is insignificant, while when k = 0.8 the effect of the value of N is perceptible. Even in this case, however, the results obtained for N = 5 and N = 10 differ by not more than 4%. Henceforth, therefore, we will confine ourselves to the value N = 10 for all k < 1.

In conclusion let us examine the dependence of the lifting force on the value of the adduced frequency of k. The coefficients of the lifting force are shown in Figure 5 on a complex plane. The solid curve is drawn through calculated points corresponding to the values k = 0, 0.2, 0.4, 0.6, and 0.8 (the values for all other parameters are the same as before). The broken curve was constructed on the basis of data given in [4]. A comparison of the solid and broken curves show that their divergence does not exceed 20%. Since both theories make use of approximate methods, which are in good agreement only for small values of k, the concurrence of the results can be considered as satisfactory.

It may thus be said that the calculation of non-stationary forces using the partitioning method described above is entirely possible and produces satisfactory results for the total lifting force, even for a relatively small number of rectangles. It should be noted that a reduction in the number of rectangles significantly reduces the volume of computational operations. But if a clearer picture of the distribution of the lifting force along the wing is required, the number of rectangles should be increased.

The effect of the wake is apparently significant only at distances of the order of the length of the chord. But in those cases where the length of the wave exceeds the length of the chord by a factor of 10 or greater the effect of the wake may be disregarded.

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